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Subsidiary Conditions of the General Gel'fand–Yaglom Wave Equation Based on the Representation $(1/2, 3/2)\oplus(-1/2, 3/2)\oplus(1/2, 5/2)\oplus(-1/2, 5/2)\oplus(1/2, 3/2)\oplus(-1/2, 3/2)$

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A method is given of obtaining the subsidiary conditions of the second kind of the general Gel'fand–Yaglom wave equation based on the representation $(1/2, 3/2)\oplus(-1/2, 3/2)\oplus(1/2, 5/2)\oplus(-1/2, 5/2)\oplus(1/2, 3/2)\oplus(-1/2, 3/2)$ and in the presence of an external electromagnetic field by reformulating the wave equation in spinor form. The wave equations accepting these subsidiary conditions form a class defined by a set of simultaneous equations that is not empty.

1. INTRODUCTION

Johnson and Sudarshan (1961) discovered that for the spin-3/2 Rarita–Schwinger (1941) wave equation the equal-time commutation relations do not vanish at spacelike points when the wave equation is minimally coupled to an external electromagnetic field. This was a serious flaw for this wave equation, since this meant that the wave equation was not causal, i.e., the velocity v of propagation of the solutions of the wave equation is greater than the speed c of light.

Thus, A.S. Wightman in 1968 (see Wightman, 1971; Velo and Wightman, 1978) proposed the investigation of the stability of relativistic wave equations.

In 1969 Velo and Zwanziger (1969a, b) took up again the question of causality of the spin-3/2 Rarita–Schwinger wave equation in the presence of an external electromagnetic field, which they studied classically using the method of characteristics (Courant and Hilbert, 1974) and they found

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that certain components of the wave equation propagate noncausally even for very weak fields.

These results stimulated studies of the causal behavior of various wave equations in the presence of various combinations of external fields (Shamaly and Capri, 1972a, b; Baisya, 1970; Nagpal, 1973; Krajcik and Nieto, 1976). Most of these works study the propagation of these wave equations using the method of characteristics. This method consists in converting the original wave equation, which has singular matrices, into a new wave equation with nonsingular matrices by using the subsidiary conditions of the second kind. Thus, in every case it is essential to be able to find these subsidiary conditions.

2. GEL'FAND–YAGLOM WAVE EQUATION

Our purpose in this paper is to give a method of finding the subsidiary conditions of the second kind of the general Gel'fand–Yaglom wave equation (Gel'fand et al., 1963)

$$\mathbb{L}_0\frac{\partial\psi}{\partial x_0}+\mathbb{L}_1\frac{\partial\psi}{\partial x_1}+\mathbb{L}_2\frac{\partial\psi}{\partial x_2}+\mathbb{L}_3\frac{\partial\psi}{\partial x_3}+i\kappa\psi=0 \tag{1}$$

based on the representation

$$(\tfrac{1}{2},\tfrac{3}{2})\oplus(-\tfrac{1}{2},\tfrac{3}{2})\oplus(\tfrac{1}{2},\tfrac{5}{2})\oplus(-\tfrac{1}{2},\tfrac{5}{2})\oplus(\tfrac{1}{2},\tfrac{3}{2})\oplus(-\tfrac{1}{2},\tfrac{3}{2}) \tag{2}$$

with components τ interlocking according to the scheme

$$\begin{tikzcd}
(\tfrac{1}{2},\tfrac{5}{2})\sim\tau_2 \arrow[rr, leftrightarrows] \arrow[dddd, leftrightarrows] \arrow[dr, leftrightarrows] & & \tau_1\sim(\tfrac{1}{2},\tfrac{3}{2}) \arrow[dddd, leftrightarrows] \\
& \tau_1'\sim(\tfrac{1}{2},\tfrac{3}{2}) \arrow[d, leftrightarrows] \arrow[dddr, leftrightarrows] & \\
& \dot{\tau}_1'\sim(-\tfrac{1}{2},\tfrac{3}{2}) \arrow[uur, leftrightarrows] & \\
& & \\
(-\tfrac{1}{2},\tfrac{5}{2})\sim\dot{\tau}_2 \arrow[rr, leftrightarrows] \arrow[uur, leftrightarrows] & & \dot{\tau}_1\sim(-\tfrac{1}{2},\tfrac{3}{2})
\end{tikzcd} \tag{3}$$

[Our notation is the same as that of Gel'fand et al. (1963).] The canonical form of the wave equation (1) with respect to the basis

$$\begin{gathered}\{\xi_{lm}\}=\{\xi^{\tau_1}_{l_1,m_1},\ \xi^{\dot{\tau}_1}_{l_1,m_1},\ \xi^{\dot{\tau}_2}_{l_1,m_1},\ \xi^{\dot{\tau}_2}_{l_2,m_2},\ \xi^{\tau_2}_{l_1,m_1},\ \xi^{\tau_2}_{l_2,m_2},\ \xi^{\tau_1'}_{l_1,m_1},\ \xi^{\dot{\tau}_1'}_{l_1,m_1}\}\\ l_1=\tfrac{1}{2},\quad m_1=\tfrac{1}{2},-\tfrac{1}{2},\quad l_2=\tfrac{3}{2},\quad m_2=\tfrac{3}{2},\tfrac{1}{2},-\tfrac{1}{2},-\tfrac{3}{2}\end{gathered} \tag{4}$$

invariant under the complete group, derivable from an invariant Lagrangian, and associated with the bilinear form (ψ_1, ψ_2) defined by the constants

$$\alpha^{\tau_1\dot{\tau}_1} = \alpha^{\dot{\tau}_1\tau_1} = 1, \qquad \alpha^{\tau_2\dot{\tau}_2} = \alpha^{\dot{\tau}_2\tau_2} = -1, \qquad \alpha^{\tau_1'\dot{\tau}_1'} = \alpha^{\dot{\tau}_1'\tau_1'} = 1 \tag{5}$$

is given in Koutroulos (1983). In particular, the matrix $\mathbb{L}_0$ has the block form

$$\mathbb{L}_0^{1/2} = \begin{array}{c} \\ \tau_1 \\ \dot{\tau}_1 \\ \dot{\tau}_2 \\ \tau_2 \\ \tau_1' \\ \dot{\tau}_1' \end{array} \begin{array}{c} \begin{array}{cccccc} \tau_1 & \dot{\tau}_1 & \dot{\tau}_2 & \tau_2 & \tau_1' & \dot{\tau}_1' \end{array} \\ \begin{pmatrix} 0 & \alpha & 0 & i\sqrt{3}\beta & 0 & \gamma \\ \alpha & 0 & i\sqrt{3}\beta & 0 & \gamma & 0 \\ 0 & i\sqrt{3}\bar{\beta} & 0 & \varepsilon & 0 & i\sqrt{3}\zeta \\ i\sqrt{3}\bar{\beta} & 0 & \varepsilon & 0 & i\sqrt{3}\zeta & 0 \\ 0 & \bar{\gamma} & 0 & i\sqrt{3}\bar{\zeta} & 0 & \theta \\ \bar{\gamma} & 0 & i\sqrt{3}\bar{\zeta} & 0 & \theta & 0 \end{pmatrix} \end{array} \tag{6}$$

$$\mathbb{L}_0^{3/2} = \begin{array}{c} \\ \dot{\tau}_2 \\ \tau_2 \end{array} \begin{array}{c} \begin{array}{cc} \dot{\tau}_2 & \tau_2 \end{array} \\ \begin{pmatrix} 0 & 2\varepsilon \\ 2\varepsilon & 0 \end{pmatrix} \end{array}$$

where α, β, γ, ε, ζ, and θ are constants. The constants α, ε, and θ are real numbers. For simplicity we introduce the new constants

$$b = i\sqrt{3}\beta, \qquad c = i\sqrt{3}\bar{\beta}, \qquad z = i\sqrt{3}\zeta, \qquad k = i\sqrt{3}\bar{\zeta} \tag{7}$$

3. SPINOR FORM OF THE GEL'FAND–YAGLOM WAVE EQUATION

We shall reformulate the Gel'fand–Yaglom wave equation in spinor language because, as we shall demonstrate below, it is then much easier to find the subsidiary conditions of the second kind which are necessary in the study of the propagation of the wave equation in the presence of an external electromagnetic field, using the method of characteristics. The subsidiary conditions of the first kind can be found easily from the canonical form of the wave equation by transforming it with the similarity transformation that converts the matrix $\mathbb{L}_0$ of the wave equation into its Jordan form and selecting those differential equations that do not involve the time derivatives, but only the space ones.

To be able to express the Gel'fand–Yaglom wave equation (1) in spinor form, it is necessary to find the similarity transformation connecting the canonical basis $\{\xi_{l,m}\}$ to the spinor basis

$$\{\alpha^{\dot{\omega}}_{\varphi\nu}, d^{\dot{\omega}}, \delta^{\dot{\omega}}, b^{\dot{\omega}\varphi}_{\nu}, c_{\omega}, \gamma_{\omega}\} \tag{8}$$

where $\dot{\omega}=\dot{1},\dot{2}$; $\varphi=1,2$; $\nu=1,2$; $\dot{\varphi}=\dot{1},\dot{2}$; $\omega=1,2$. This transformation can be found as follows. If

$$\mathbb{H}_3^c, \mathbb{H}_+^c, \mathbb{H}_-^c, \mathbb{F}_3^c, \mathbb{F}_+^c, \mathbb{F}_-^c, \mathbb{L}_j^c \qquad (j=0,1,2,3) \tag{9}$$

are the generators and matrices of the wave equation in the canonical frame and

$$\mathbb{H}_3^s, \mathbb{H}_+^s, \mathbb{H}_-^s, \mathbb{F}_3^s, \mathbb{F}_+^s, \mathbb{F}_-^s, \mathbb{L}_j^s \tag{10}$$

are the generators and matrices of the wave equation in the spinor basis, then the similarity transformation $\mathbb{T}$ must be such tht it satisfies the relations

$$\begin{gathered}
\mathbb{T}\mathbb{H}_3^c\mathbb{T}^{-1}=\mathbb{H}_3^s, \qquad \mathbb{T}\mathbb{H}_+^c\mathbb{T}^{-1}=\mathbb{H}_+^s, \qquad \mathbb{T}\mathbb{H}_-^c\mathbb{T}^{-1}=\mathbb{H}_-^s \\
\mathbb{T}\mathbb{F}_3^c\mathbb{T}^{-1}=\mathbb{F}_3^s, \qquad \mathbb{T}\mathbb{F}_+^c\mathbb{T}^{-1}=\mathbb{F}_+^s, \qquad \mathbb{T}\mathbb{F}_-^c\mathbb{T}^{-1}=\mathbb{F}_-^s \\
\mathbb{T}\mathbb{L}_j^c\mathbb{T}^{-1}=\mathbb{L}_j^s
\end{gathered} \tag{11}$$

These relations are sufficient to determine $\mathbb{T}$. Thus, finding $\mathbb{T}$, it can be shown that the general 20-dimensional Gel'fand–Yaglom wave equation for maximum spin 3/2 based on the representation (2) invariant under the complete group derivable from an invariant Lagrangian and associated with the bilinear form (5) in the presence of an external electromagnetic field acquires the following spinor form:

$$-\pi_{1\dot{1}}b_1^{\dot{1}\dot{1}}-\pi_{1\dot{2}}b_1^{\dot{1}\dot{2}}+2C\pi_1^{\dot{1}}c_1+2Z\pi_1^{\dot{1}}\gamma_1+\chi\alpha_{11}^{\dot{1}}=0 \tag{E1}$$

$$\begin{aligned}
&-\tfrac{1}{2}\pi_{2\dot{2}}b_1^{\dot{1}\dot{2}}-\tfrac{1}{2}\pi_{1\dot{1}}b_2^{\dot{1}\dot{1}}+C\pi_2^{\dot{1}}c_1+Z\pi_2^{\dot{1}}\gamma_1-\tfrac{1}{2}\pi_{12}b_1^{\dot{1}\dot{1}}\\
&\quad-\tfrac{1}{2}\pi_{1\dot{2}}b_2^{\dot{1}\dot{2}}+C\pi_1^{\dot{1}}c_2+Z\pi_1^{\dot{1}}\gamma_2+\chi\alpha_{12}^{\dot{1}}=0
\end{aligned} \tag{E2}$$

$$-\pi_{2\dot{2}}b_2^{\dot{1}\dot{2}}+2C\pi_2^{\dot{1}}c_2+2Z\pi_2^{\dot{1}}\gamma_2-\pi_{2\dot{1}}b_2^{\dot{1}\dot{1}}+\chi\alpha_{22}^{\dot{1}}=0 \tag{E3}$$

$$-\pi_{1\dot{1}}b_1^{\dot{1}\dot{2}}+2C\pi_1^{\dot{2}}c_1+2Z\pi_1^{\dot{2}}\gamma_1-\pi_{1\dot{2}}b_1^{\dot{2}\dot{2}}+\chi\alpha_{11}^{\dot{2}}=0 \tag{E4}$$

$$\begin{aligned}
&-\tfrac{1}{2}\pi_{\dot{2}2}b_1^{\dot{2}\dot{2}}-\tfrac{1}{2}\pi_{1\dot{1}}b_2^{\dot{1}\dot{2}}+C\pi_1^{\dot{2}}c_2+Z\pi_1^{\dot{2}}\gamma_2-\tfrac{1}{2}\pi_{2\dot{1}}b_1^{\dot{1}\dot{2}}\\
&\quad-\tfrac{1}{2}\pi_{1\dot{2}}b_2^{\dot{2}\dot{2}}+C\pi_2^{\dot{2}}c_1+Z\pi_2^{\dot{2}}\gamma_1+\chi\alpha_{12}^{\dot{2}}=0
\end{aligned} \tag{E5}$$

$$-\pi_{2\dot{2}}b_2^{\dot{2}\dot{2}}-\pi_{2\dot{1}}b_2^{\dot{1}\dot{2}}+2C\pi_2^{\dot{2}}c_2+2Z\pi_2^{\dot{2}}\gamma_2+\chi\alpha_{22}^{\dot{2}}=0 \tag{E6}$$

$$\begin{aligned}
&-\tfrac{1}{3}B\pi^1_{\dot{2}}b_1^{\dot{1}\dot{2}}-\tfrac{1}{3}B\pi^2_{\dot{1}}b_2^{\dot{1}\dot{1}}-A\pi^{1\dot{1}}c_1-\Gamma\pi^{1\dot{1}}\gamma_1-\tfrac{1}{3}\pi^1_{\dot{1}}b_1^{\dot{1}\dot{1}}\\
&\quad-\tfrac{1}{3}B\pi^2_{\dot{2}}b_2^{\dot{1}\dot{2}}-A\pi^{\dot{1}2}c_2-\Gamma\pi^{\dot{1}2}\gamma_2+\chi d^{\dot{1}}=0
\end{aligned} \tag{E7}$$

$$\begin{aligned}
&-\tfrac{1}{3}B\pi^1_{\dot{2}}b_1^{\dot{2}\dot{2}}-\tfrac{1}{3}B\pi^2_{\dot{1}}b_2^{\dot{1}\dot{2}}-A\pi^{\dot{2}2}c_2-\Gamma\pi^{\dot{2}2}\gamma_2-\tfrac{1}{3}B\pi^1_{\dot{1}}b_1^{\dot{1}\dot{2}}\\
&\quad-\tfrac{1}{3}B\pi^2_{\dot{2}}b_2^{\dot{2}\dot{2}}-A\pi^{\dot{2}1}c_1-\Gamma\pi^{\dot{2}1}\gamma_1+\chi d^{\dot{2}}=0
\end{aligned} \tag{E8}$$

$$\begin{aligned}
&-\tfrac{1}{3}K\pi^1_{\dot{2}}b_1^{\dot{1}\dot{2}}-\tfrac{1}{3}K\pi^2_{\dot{1}}b_2^{\dot{1}\dot{1}}-\bar{\Gamma}\pi^{1\dot{1}}c_1-\Theta\pi^{1\dot{1}}\gamma_1-\tfrac{1}{3}K\pi^1_{\dot{1}}b_1^{\dot{1}\dot{1}}\\
&\quad-\tfrac{1}{3}K\pi^2_{\dot{2}}b_2^{\dot{1}\dot{2}}-\bar{\Gamma}\pi^{\dot{1}2}c_2-\Theta\pi^{\dot{1}2}\gamma_2+\chi\delta^{\dot{1}}=0
\end{aligned} \tag{E9}$$

$$-\tfrac{1}{3}K\pi^{1}_{\dot2}b_1^{\dot2\dot2}-\tfrac{1}{3}K\pi^{2}_{\dot1}b_2^{\dot1\dot2}-\bar\Gamma\pi^{\dot22}c_2-\Theta\pi^{\dot22}\gamma_2-\tfrac{1}{3}K\pi^{1}_{\dot1}b_1^{\dot1\dot2}$$
$$-\tfrac{1}{3}K\pi^{2}_{\dot2}b_2^{\dot2\dot2}-\bar\Gamma\pi^{\dot21}c_1-\Theta\pi^{\dot21}\gamma_1+\chi\delta^{\dot2}=0 \tag{E10}$$
$$-\pi^{\dot11}\alpha^{\dot1}_{11}-\pi^{\dot12}\alpha^{\dot1}_{12}+2C\pi^{\dot1}_{1}d^{\dot1}+2Z\pi^{\dot1}_{1}\delta^{\dot1}+\chi b_1^{\dot1\dot1}=0 \tag{E11}$$
$$-\tfrac{1}{2}\pi^{\dot22}\alpha^{\dot1}_{22}-\tfrac{1}{2}\pi^{\dot11}\alpha^{\dot2}_{11}+C\pi^{\dot2}_{1}d^{\dot1}+Z\pi^{\dot2}_{1}\delta^{\dot1}-\tfrac{1}{2}\pi^{\dot21}\alpha^{\dot1}_{11}$$
$$-\tfrac{1}{2}\pi^{\dot12}\alpha^{\dot2}_{12}+C\pi^{\dot1}_{1}d^{\dot2}+Z\pi^{\dot1}_{1}\delta^{\dot2}+\chi b_1^{\dot1\dot2}=0 \tag{E12}$$
$$-\pi^{\dot22}\alpha^{\dot2}_{12}+2C\pi^{\dot2}_{1}d^{\dot2}+2Z\pi^{\dot2}_{1}\delta^{\dot2}-\pi^{\dot21}\alpha^{\dot2}_{11}+\chi b_1^{\dot2\dot2}=0 \tag{E13}$$
$$-\pi^{\dot11}\alpha^{\dot1}_{12}+2C\pi^{\dot1}_{2}d^{\dot1}+2Z\pi^{\dot1}_{2}\delta^{\dot1}-\pi^{\dot12}\alpha^{\dot1}_{22}+\chi b_2^{\dot1\dot1}=0 \tag{E14}$$
$$-\tfrac{1}{2}\pi^{\dot22}\alpha^{\dot1}_{22}-\tfrac{1}{2}\pi^{\dot11}\alpha^{\dot2}_{12}+C\pi^{\dot1}_{2}d^{\dot2}+Z\pi^{\dot1}_{2}\delta^{\dot2}-\tfrac{1}{2}\pi^{\dot21}\alpha^{\dot1}_{12}$$
$$-\tfrac{1}{2}\pi^{\dot12}\alpha^{\dot2}_{22}+C\pi^{\dot2}_{2}d^{\dot1}+Z\pi^{\dot2}_{2}\delta^{\dot1}+\chi b_2^{\dot1\dot2}=0 \tag{E15}$$
$$-\pi^{\dot22}\alpha^{\dot2}_{22}-\pi^{\dot21}\alpha^{\dot2}_{12}+2C\pi^{\dot2}_{2}d^{\dot2}+2Z\pi^{\dot2}_{2}\delta^{\dot2}+\chi b_2^{\dot2\dot2}=0 \tag{E16}$$
$$-\tfrac{1}{3}B\pi^{2}_{\dot1}\alpha^{\dot1}_{12}-\tfrac{1}{3}B\pi^{1}_{\dot2}\alpha^{\dot2}_{11}-A\pi_{\dot11}d^{\dot1}-\Gamma\pi_{\dot11}\delta^{\dot1}-\tfrac{1}{3}B\pi^{1}_{\dot1}\alpha^{\dot1}_{11}$$
$$-\tfrac{1}{3}B\pi^{2}_{\dot2}\alpha^{\dot2}_{12}-A\pi_{1\dot2}d^{\dot2}-\Gamma\pi_{1\dot2}\delta^{\dot2}+\chi c_1=0 \tag{E17}$$
$$-\tfrac{1}{3}B\pi^{2}_{\dot1}\alpha^{\dot1}_{22}-\tfrac{1}{3}B\pi^{1}_{\dot2}\alpha^{\dot2}_{12}-A\pi_{\dot22}d^{\dot2}-\Gamma\pi_{\dot22}\delta^{\dot2}-\tfrac{1}{3}B\pi^{1}_{\dot1}\alpha^{\dot1}_{12}$$
$$-\tfrac{1}{3}B\pi^{2}_{\dot2}\alpha^{\dot2}_{22}-A\pi_{2\dot1}d^{\dot1}-\Gamma\pi_{2\dot1}\delta^{\dot1}+\chi c_2=0 \tag{E18}$$
$$-\tfrac{1}{3}K\pi^{2}_{\dot1}\alpha^{\dot1}_{12}-\tfrac{1}{3}K\pi^{1}_{\dot2}\alpha^{\dot2}_{11}-\bar\Gamma\pi_{\dot11}d^{\dot1}-\Theta\pi_{\dot11}\delta^{\dot1}-\tfrac{1}{3}K\pi^{1}_{\dot1}\alpha^{\dot1}_{11}$$
$$-\tfrac{1}{3}K\pi^{2}_{\dot2}\alpha^{\dot2}_{12}-\bar\Gamma\pi_{1\dot2}d^{\dot2}-\Theta\pi_{1\dot2}\delta^{\dot2}+\chi\gamma_1=0 \tag{E19}$$
$$-\tfrac{1}{3}K\pi^{2}_{\dot1}\alpha^{\dot1}_{22}-\tfrac{1}{3}K\pi^{1}_{\dot2}\alpha^{\dot2}_{12}-\bar\Gamma\pi_{\dot22}d^{\dot2}-\Theta\pi_{\dot22}\delta^{\dot2}-\tfrac{1}{3}K\pi^{1}_{\dot1}\alpha^{\dot1}_{12}$$
$$-\tfrac{1}{3}K\pi^{2}_{\dot2}\alpha^{\dot2}_{22}-\bar\Gamma\pi_{2\dot1}d^{\dot1}-\Theta\pi_{2\dot1}\delta^{\dot1}+\chi\gamma_2=0 \tag{E20}$$

where C, Z, B, A, Γ, K, and Θ are constants related to the constants, c, z, b, α, γ, k, and θ entering the matrix $\mathbb{L}_0$ by the relations

$$C=\frac{c}{2\varepsilon},\qquad Z=\frac{z}{2\varepsilon},\qquad B=\frac{b}{2\varepsilon},\qquad A=\frac{\alpha}{2\varepsilon},\qquad \Gamma=\frac{\gamma}{2\varepsilon},$$
$$K=\frac{k}{2\varepsilon},\qquad \Theta=\frac{\theta}{2\varepsilon} \tag{12}$$

which amounts to dividing the wave equation (1) throughout by 2ε. Note that by dividing throughout by 2ε we are restricting the block $\mathbb{L}_0^{3/2}$ to having nonvanishing eigenvalues, and hence the wave equation will describe spin-3/2 particles with or without spin-1/2 particles present, depending on the eigenvalues of the block $\mathbb{L}_0^{1/2}$. A bar above a quantity indicates its complex conjugate. $\pi_{\sigma\dot\rho}$ $(\sigma=1,2;\ \dot\rho=\dot1,\dot2)$ are the electromagnetic spinor components connected to the four momentum electromagnetic vector components π_r $(r=0,1,2,3)$ by the formulas

$$\pi_{\dot11}=-\pi_0+\pi_3,\qquad \pi_{\dot21}=\pi_1+i\pi_2,\qquad \pi_{1\dot2}=\pi_1-i\pi_2,\qquad \pi_{\dot22}=-\pi_0-\pi_3 \tag{13}$$

$\chi=\kappa/2\varepsilon$ is a constant related to the masses of the particles associated with the field. $i=\sqrt{-1}$. (The free field wave equation follows from the above one for zero electromagnetic four-vector potential.)

We notice that if the constants (12) have the values

$$B=\tfrac{1}{2},\qquad C=-\tfrac{1}{2},\qquad A=-\tfrac{1}{2},\qquad Z=K=\Theta=\Gamma=0 \tag{14}$$

then the above wave equation goes over into the spinor form of the Pauli-Fierz wave equation for spin 3/2 (Fierz and Pauli, 1939; Gupta, 1954).

4. SUBSIDIARY CONDITIONS

We next find the class of all those 20-dimensional wave equations with maximum spin-3/2 accepting subsidiary conditions of the second kind. For this let us multiply equation (E11) by $\lambda\pi_{\dot{1}}^{1}$, (E12) by $\lambda\pi_{\dot{2}}^{1}$, (E14) by $\lambda\pi_{\dot{1}}^{2}$, and (E15) by $\lambda\pi_{\dot{2}}^{2}$ and add,

$$\lambda\pi_{\dot{1}}^{1}\times(\mathrm{E}11)+\lambda\pi_{\dot{2}}^{1}\times(\mathrm{E}12)+\lambda\pi_{\dot{1}}^{2}\times(\mathrm{E}14)+\lambda\pi_{\dot{2}}^{2}\times(\mathrm{E}15)=0 \tag{15}$$

where λ is a constant to be determined. Similarly, let us multiply (E17) by $3\xi\pi^{1\dot{1}}$, (E18) by $3\xi\pi^{2\dot{1}}$, (E19) by $3\xi\pi^{1\dot{1}}$, and (E20) and $3\xi\pi^{2\dot{1}}$ and add,

$$3\xi\pi^{1\dot{1}}\times(\mathrm{E}17)+3\xi\pi^{2\dot{1}}\times(\mathrm{E}18)+3\xi\pi^{1\dot{1}}\times(\mathrm{E}19)+3\xi\pi^{2\dot{1}}\times(\mathrm{E}20)=0 \tag{16}$$

where ξ is a constant to be determined. Subtracting (16) from (15), we have

$$\begin{aligned}&\lambda\{\pi_{\dot{1}}^{1}\times(\mathrm{E}11)+\pi_{\dot{2}}^{1}\times(\mathrm{E}12)+\pi_{\dot{1}}^{2}\times(\mathrm{E}14)+\pi_{\dot{2}}^{2}\times(\mathrm{E}15)\}\\&\quad-3\xi\{\pi^{1\dot{1}}(\mathrm{E}17)+\pi^{2\dot{1}}\times(\mathrm{E}18)+\pi^{1\dot{1}}\times(\mathrm{E}19)+\pi^{2\dot{1}}\times(\mathrm{E}20)\}=0\end{aligned} \tag{17}$$

Substituting into (17)

$$-6\chi\times(\mathrm{E}7)-\mu6\chi\times(\mathrm{E}9) \tag{18}$$

(where μ is a constant to be determined), having imposed the conditions

$$2B+2K\mu=\lambda,\qquad 2A+2\bar{\Gamma}\mu=-\xi,\qquad 2\Gamma+2\Theta\mu=-\xi \tag{19}$$

replacing $\pi_{\sigma\dot{\rho}}$ by the relations (13), and imposing the condition

$$-\tfrac{1}{2}\lambda+(B+K)\xi=0 \tag{20}$$

in order to create terms involving $[\pi_p,\pi_q]=ieF_{pq}=f_{pq}$ (where F_{pq}, $p,q=0,1,2,3$, is the electromagnetic tensor), and finally imposing the conditions

$$3\lambda C-3\xi(A+\bar{\Gamma})=0,\qquad 3\lambda Z-3\xi(\Gamma+\Theta)=0 \tag{21}$$

in order to make terms involving $(\pi_r)^2$, $r=0,1,2,3$, vanish, we obtain the following subsidiary condition of the second kind:

$$\begin{aligned}&6\chi^2d^{\dot{1}}+6\mu\chi^2\delta^{\dot{1}}+\lambda(f_{10}+f_{13}+if_{32}+if_{02})\alpha_{11}^{\dot{1}}+2\lambda(f_{03}+if_{12})\alpha_{12}^{\dot{1}}\\&\quad+[\lambda C+3\xi(A+\bar{\Gamma})](if_{12}+f_{30})d^{\dot{1}}+[\lambda Z+3\xi(\Gamma+\Theta)](if_{12}+f_{30})\delta^{\dot{1}}\\&\quad+[\lambda C+3\xi(A+\bar{\Gamma})](f_{10}+if_{20}+f_{13}+if_{23})d^{\dot{2}}+[\lambda Z+3\xi(\Gamma+\Theta)]\\&\quad\times(f_{10}+f_{13}+if_{20}+if_{23})\delta^{\dot{2}}+\lambda(f_{01}+if_{02}+if_{23}+f_{13})\alpha_{22}^{\dot{1}}=0\end{aligned} \tag{22}$$

By similar operations to the above, three more subsidiary conditions of the second kind can be obtained. For the second subsidiary condition we perform the following operations:

$$\lambda\{\pi_1^1 \times (E_{12}) + \pi_2^1 \times (E13) + \pi_1^2 \times (E15) + \pi_2^2 \times (E16)\}$$
$$-3\xi\{\pi^{1\dot{2}} \times (E17) + \pi^{2\dot{2}} \times (E18) + \pi^{1\dot{2}} \times (E19) + \pi^{2\dot{2}} \times (E20)\} = 0 \quad (23)$$

Substituting into this relation the expression

$$-6\chi \times (E8) - \mu 6\chi \times (E10) \quad (24)$$

and imposing the conditions (19)-(21), we obtain the subsidiary condition

$$6\chi^2 d^{\dot{2}} + \mu 6\chi^2 \delta^{\dot{2}} + \lambda(f_{10} + f_{13} + if_{02} + if_{32})\alpha_{11}^{\dot{2}}$$
$$+ [\lambda C + 3\xi(A + \bar{\Gamma})](f_{10} + if_{02} + f_{31} + if_{23})d^{\dot{1}}$$
$$+ [\lambda Z + 3\xi(\Gamma + \Theta)](f_{10} + if_{02} + f_{31} + if_{23})\delta^{\dot{1}}$$
$$+ 2\lambda(if_{12} + f_{03})\alpha_{12}^{\dot{2}} + [\lambda C + 3\xi(A + \bar{\Gamma})](if_{21} + f_{03})d^{\dot{2}}$$
$$+ [\lambda Z + 3\xi(\Gamma + \Theta)](if_{21} + f_{03})\delta^{\dot{2}}$$
$$+ \lambda(f_{01} + if_{02} + f_{13} + if_{23})\alpha_{12}^{\dot{2}} = 0 \quad (25)$$

For the third subsidiary condition we perform the following operations:

$$\lambda\{\pi_1^1 \times (E1) + \pi_1^2 \times (E2) + \pi_2^1 \times (E4) + \pi_2^2 \times (E5)\}$$
$$-3\xi\{\pi_{\dot{1}1} \times (E7) + \pi_{\dot{2}1} \times (E8) + \pi_{\dot{1}1} \times (E9) + \pi_{\dot{2}1} \times (E_{10})\} = 0 \quad (26)$$

Substituting into this relation the expression

$$-6\chi \times (E17) - \mu 6\chi \times (E19)$$

and imposing again the conditions (19)-(21), we obtain the subsidiary condition

$$-6\chi^2 c_1 + \mu 6\chi^2 \gamma_1 + \lambda(f_{10} + f_{31} + if_{02} + if_{23})b_1^{\dot{1}\dot{1}} + 2\lambda(if_{21} + f_{03})b_1^{\dot{1}\dot{2}}$$
$$+ [\lambda C + 3\xi(A + \bar{\Gamma})](if_{12} + f_{03})c_1$$
$$+ [\lambda Z + 3\xi(\Gamma + \Theta)](if_{12} + f_{03})\gamma_1$$
$$+ [(\lambda C + 3\xi(A + \bar{\Gamma})](f_{01} + if_{02} + if_{23} + f_{13})c_2$$
$$+ [\lambda Z + 3\xi(\Gamma + \Theta)](f_{01} + if_{02} + f_{13} + if_{23})\gamma_2$$
$$+ \lambda(f_{01} + if_{02} + f_{31} + if_{32})b_1^{\dot{2}\dot{2}} = 0 \quad (27)$$

For the fourth subsidiary condition we perform the following operations:

$$\begin{aligned}\lambda\{\pi_{1}^{1}\times(\mathrm{E2})+\pi_{1}^{2}\times(\mathrm{E3})+\pi_{2}^{1}\times(\mathrm{E5})+\pi_{2}^{2}\times(\mathrm{E6})\}\\ -3\xi\{\pi_{2\mathrm{i}}\times(\mathrm{E7})+\pi_{\dot{2}2}\times(\mathrm{E8})+\pi_{2\mathrm{i}}\times(\mathrm{E9})+\pi^{2\dot{2}}\times(\mathrm{E10})\}=0\end{aligned} \tag{28}$$

Substituting into this relation the expression

$$-6\chi\times(\mathrm{E18})-\mu 6\chi\times(\mathrm{E20}) \tag{29}$$

and imposing the conditions (19)-(21) we obtain the subsidiary condition

$$\begin{aligned}&6\chi^2c_2+6\mu\chi^2\gamma_2+\lambda(f_{10}+f_{31}+if_{02}+if_{23})b_2^{\mathrm{ii}}\\ &+[\lambda C+3\xi(A+\bar{\Gamma})](f_{01}+if_{02}+f_{31}+if_{23})c_1\\ &+[\lambda Z+3\xi(\Gamma+\Theta)](f_{01}+f_{31}+if_{20}+if_{23})\gamma_1\\ &-2\lambda(f_{03}+if_{21})b_2^{\mathrm{i}\dot{2}}+[\lambda C+3\xi(A+\bar{\Gamma})](if_{21}+f_{30})c_2\\ &+[\lambda Z+3\xi(\Gamma+\Theta)](if_{21}+f_{30})\gamma_2\\ &+\lambda(f_{01}+if_{02}+f_{31}+if_{32})b_2^{\dot{2}\dot{2}}=0\end{aligned} \tag{30}$$

Thus, if the constants entering the general 20-dimensional wave equation (1) are such that the relations (19)-(21) are satisfied simultaneously, then the wave equation accepts subsidiary conditions of the second kind given above, i.e., involving the field components f_{pq}. These relations define a class of 20-dimensional wave equations that is not empty. Examples belonging to it are given below.

EXAMPLES

We give now examples of wave equations for which one can find the subsidary conditions of the second kind as described above.

Example 1. If the constants ξ, μ, and λ have the values $\xi=1$, $\mu=1$, and $\lambda=\sqrt{2}$ and the constants entering the wave equation have the values

$$\begin{aligned}&B=\frac{1}{2\sqrt{2}},\quad C=-\frac{1}{2\sqrt{2}},\quad Z=-\frac{1}{2\sqrt{2}},\quad K=\frac{1}{2\sqrt{2}},\quad A=-\frac{1}{4}\\ &\Gamma=-\frac{1}{4},\quad \Theta=-\frac{1}{4},\quad \chi\neq 0\end{aligned} \tag{31}$$

the set of simultaneous equations (19)-(21) is satisfied and a spin-3/2 Gel'fand-Yaglom wave equation is defined whose matrix $\mathbb{L}_0$ has in the canonical basis the blocks

$$\mathbb{L}_0^{1/2} = \begin{array}{c} \\ \tau_1 \\ \dot{\tau}_1 \\ \dot{\tau}_2 \\ \tau_2 \\ \tau_1' \\ \dot{\tau}_1' \end{array} \begin{array}{c} \begin{array}{cccccc} \tau_1 & \dot{\tau}_1 & \dot{\tau}_2 & \tau_2 & \tau_1' & \dot{\tau}_1' \end{array} \\ \begin{pmatrix} 0 & -\frac{1}{4} & 0 & 1/2\sqrt{2} & 0 & -\frac{1}{4} \\ -\frac{1}{4} & 0 & 1/2\sqrt{2} & 0 & -\frac{1}{4} & 0 \\ 0 & -1/2\sqrt{2} & 0 & \frac{1}{2} & 0 & -1/2\sqrt{2} \\ -1/2\sqrt{2} & 0 & \frac{1}{2} & 0 & -1/2\sqrt{2} & 0 \\ 0 & -\frac{1}{4} & 0 & 1/2\sqrt{2} & 0 & -\frac{1}{4} \\ -\frac{1}{4} & 0 & 1/2\sqrt{2} & 0 & -\frac{1}{4} & 0 \end{pmatrix} \end{array} \tag{32}$$

$$\mathbb{L}_0^{3/2} = \begin{array}{c} \\ \dot{\tau}_2 \\ \tau_2 \end{array} \begin{array}{c} \begin{array}{cc} \dot{\tau}_2 & \tau_2 \end{array} \\ \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \end{array}$$

Notice that the eigenvalues of the block $\mathbb{L}_0^{1/2}$ are all zero and hence the corresponding wave equation describes spin-3/2 particles. The matrix $\mathbb{L}_0$ satisfies the minimal equation

$$\mathbb{L}_0^2[\mathbb{L}_0^2 - 1] = 0 \tag{33}$$

The charge associated with the wave equation is definite. Notice that the matrix $\mathbb{L}_0$ has $\det \mathbb{L}_0 = 0$, i.e., is singular.

Example 2. A second example of a 20-dimensional Gel'fand–Yaglom wave equation accepting subsidiary conditions of the second kind arises if the constants have the values

$$\xi = 1, \qquad \mu = 1, \qquad \lambda = \sqrt{2}, \qquad B = \frac{1}{2\sqrt{2}}, \qquad C = -\frac{1}{2\sqrt{2}}, \qquad Z = -\frac{1}{2\sqrt{2}}$$

$$K = \frac{1}{2\sqrt{2}}, \qquad A = \Theta = 0, \qquad \Gamma = -\frac{1}{2} \tag{34}$$

In this case the resulting wave equation describes spin-3/2 particles together with spin-1/2 particles (since the eigenvalues of the block $\mathbb{L}_0^{1/2}$ are not all zero). $\mathbb{L}_0$ satisfies the minimal equation

$$\mathbb{L}_0^2[\mathbb{L}_0^2 - (\tfrac{1}{2})^2] \cdot [\mathbb{L}_0^2 - 1] = 0 \tag{35}$$

Example 3. A third example occurs if the constants have the values

$$\xi = 1, \qquad \mu = 1, \qquad \lambda = \sqrt{2}, \qquad B = \frac{1}{2\sqrt{2}}, \qquad C = -\frac{1}{2\sqrt{2}}, \qquad Z = -\frac{1}{2\sqrt{2}}$$

$$K = \frac{1}{2\sqrt{2}}, \qquad A = -\frac{1}{2}, \qquad \Theta = \frac{1}{2}, \qquad \Gamma = 0 \tag{36}$$

This example again describes two kinds of particles, namely spin-3/2 and spin-1/2 particles. $\mathbb{L}_0$ satisfies the minimal equation

$$\mathbb{L}_0^2[\mathbb{L}_0^2-(\tfrac{1}{2})^2]\cdot[\mathbb{L}_0^2-1]=0 \tag{37}$$

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